

# CARLEMAN-VEKUA EQUATION WITH A SINGULAR POINT

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**Abstract.** In this article unconditional solvability of the Carleman-Vekua equation with a singular point is proved, the Riemann-Hilbert problem is solved, integral representations of solutions, the structures of their zeros and poles are received.

**Keywords:** Carleman - Vekua equation; complex plane; singular point; Riemann-Hilbert boundary value problem; holomorphic functions.

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## Introduction.

Let  $G$  be a bounded domain of the complex plane  $\mathbb{C}$  with boundary  $\Gamma \in C^{1,\alpha}$ ,  $0 < \alpha < 1$ , with the inner point  $z = a$ .

Let  $S(G)$  be the set of measurable, essentially bounded functions  $f(z)$  in  $G$  with the norm

$$\|f\|_{S(G)} = \operatorname{ess\,sup}_{z \in G} |f(z)| = \lim_{p \rightarrow \infty} \|f\|_{L_p(G)}.$$

Now the spaces used below are defined:

$S_\nu(G, a)$  is the class of functions  $f(z)$ , for which  $f(z)|z - a|^\nu \in S(G)$ . The norm in  $S_\nu(G, a)$  is defined by the formula

$$\|f\|_{S_\nu(G, a)} = \operatorname{ess\,sup}_{z \in G} (|z - a|^\nu |f(z)|),$$

where  $\nu$  is a real number.

$C_\nu(\overline{G}, a)$  is the class of functions  $f(z)$ , for which  $f(z)|z - a|^\nu \in C(\overline{G})$ . The norm in  $C_\nu(\overline{G}, a)$  is defined by the formula

$$\|f\|_{C_\nu(\overline{G}, a)} = \max_{z \in \overline{G}} (|z - a|^\nu |f(z)|).$$

$U_0(G)$  is the class of holomorphic functions in  $G$ .

$W_p^1(G)$ ,  $p > 1$  is the Sobolev space, see [1].

Let us consider the equation

$$\partial_{\bar{z}} V + A(z)V + B(z)\overline{V} = F(z), \quad (1)$$

in  $G$ , where  $\partial_{\bar{z}} = \frac{1}{2} \left( \frac{\partial}{\partial x} + i \frac{\partial}{\partial y} \right)$ ;  $A(z), B(z) \in S_1(G, a)$ ;  $F(z) \in S_\beta(G, a)$ ,  $\max\{0, 1 - 8\mu\} < \beta < \frac{2}{q}$ . Here  $\mu = \|A\|_{S_1(G, a)} + \|B\|_{S_1(G, a)}$ ;  $2 < q < \frac{2}{1-8\mu}$  if  $\mu < \frac{1}{8}$  and  $q > 2$  if  $\mu \geq \frac{1}{8}$ .

For  $a = 0$  we receive the equation

$$\partial_{\bar{z}} V + \frac{A_0(z)}{|z|} V + \frac{B_0(z)}{|z|} \overline{V} = F(z), \quad (2)$$

where  $A_0(z), B_0(z) \in S(G)$ ;  $F(z) \in S_\beta(G, 0)$ .

It is obvious, that  $F(z) \in L_q(G)$ ,  $q > 2$ .

The equation (2) with  $F(z) \equiv 0$  arise in the theory of infinitesimal deformations of surfaces of positive curvature with a flat point, see [2, 3]. There it is required to prove the existence of continuous solutions of equation (2) in the neighborhood of a singular point  $z = 0$ . In this connection the equation (2) is studied in many works of L.G. Mikhailov, Z.D Usmanov, A. Tungatarov, see [4-7], etc.

In all results usually a sufficient smallness of the coefficients  $A_0(z)$  and  $B_0(z)$  or the smallness of the domain  $G$  is supposed. In the present article we prove existence of solutions  $V(z)$  of the equation (2), satisfying to the condition  $V(0) = 0$ , without any conditions on the smallness of the coefficients or on the smallness of the domain  $G$ . Integral representations of solutions of equation (1) in the class

$$W_q^1(G) \cap C_{\beta-1}(\overline{G}, a), 0 < \beta < \frac{2}{q}, q > 2 \quad (3)$$

are also received. The Riemann – Hilbert problem for the equation (2) is solved in the class (3) without any smallness conditions on the coefficients assumed in [4-7]. The structures of the zeros and poles of the solutions of the equation (1) in the class (3) are investigated.

### §1. Representations of solutions and solvability of the equation (1)

The solutions of the equation (1) are looked for in the class (3). As in [1] it is possible to prove, that the equation (1) is equivalent to the equation

$$V(z) + (P_G V)(z) = (T_G F)(z) + \Phi(z), \quad z \in G, \quad (4)$$

where

$$(T_G f)(z) = -\frac{1}{\pi} \iint_G \frac{f(\zeta)}{\zeta - z} dG_\zeta, \quad (dG_\zeta = d\xi d\eta, \quad \zeta = \xi + i\eta),$$

$$\Phi(z) \in U_0(G), \quad (P_G f)(z) = (T_G f^*)(z), \quad f^* = A(z)f + B(z)\bar{f}.$$

As the function  $V(z)$  belongs to the class (3), then  $V(a) = 0$ . Therefore from (4) we receive

$$(P_G V)(a) = (T_G F)(a) + \Phi(a). \quad (5)$$

From (4) and (5) it follows

$$V(z) + (P_{G,a} V)(z) = (T_{G,a} F)(z) + (z - a)\Phi(z), \quad z \in G, \quad (6)$$

where

$$(P_{G,a} f)(z) = (P_G f)(z) - (P_G f)(a),$$

$$(T_{G,a} f)(z) = (T_G f)(z) - (T_G f)(a), \quad \Phi(z) \in U_0(G).$$

Thus, any solution of equation (1) from the class (3) satisfies the equation (6). A solution of the equation (6) is looked for in the class  $C_{\beta-1}(\overline{G}, a)$ . Let us show, that any solution of the equation

(6) from the class  $C_{\beta-1}(\overline{G}, a)$  belongs to the class (3) and almost everywhere in  $G$  satisfies the equation (1). Hereafter  $M$  denotes positive constants, not depending on the factor involved.

**Lemma 1.** *The operator  $(T_{G,af})(z)$  maps the class  $S_\beta(G, a)$ ,  $0 < \beta < \frac{2}{q}$ ,  $q > 2$  into the class  $C_{\beta-1}(\overline{G}, a) \cap C^\alpha(\overline{G})$ ,  $\alpha = 1 - \frac{2}{q}$ ,  $0 < \beta < \frac{2}{q}$ ,  $q > 2$  and moreover, the estimates*

$$\|(T_{G,af})(z)\|_{G_{\beta-1}(\overline{G}, a)} \leq M_\beta(G) \|f\|_{S_\beta(G, a)}, \quad (7)$$

$$\|(T_{G,af})(z)\|_{C^\alpha(\overline{G})} \leq M \|f\|_{L_q(G)} \quad (8)$$

are hold, where

$$M_\beta(G) = \sup_{z \in G} \frac{|z - a|^\beta}{\pi} \iint_G \frac{dG_\zeta}{|\zeta - a|^{1+\beta} |\zeta - z|}.$$

**Proof.** Let  $f(z) \in S_\beta(G, a)$ ,  $0 < \beta < \frac{2}{q}$ ,  $q > 2$ . Using the Hadamard inequality, we have

$$|(T_{G,af})(z)| \leq M_\beta(G) \|f\|_{S_\beta(G, a)} |z - a|^{1-\beta}.$$

From here the inequality (7) follows. As  $f(z) \in L_q(G)$ ,  $q > 2$ , the inequality (8) follows from the estimation (6.8) of the work [1, ch.1, §6]. The lemma is proved.

**Lemma 2.** *The operator  $(P_{G,af})(z)$  maps the space  $C_{\beta-1}(\overline{G}, a)$  into the space  $C_{\beta-1}(\overline{G}, a) \cap C^\alpha(\overline{G})$ ,  $\alpha = 1 - \frac{2}{q}$ ,  $0 < \beta < \frac{2}{q}$ ,  $q > 2$ , and moreover, the estimates*

$$\|(P_{G,af})(z)\|_{G_{\beta-1}(\overline{G}, a)} \leq \mu M_\beta(G) \|f\|_{C_{\beta-1}(\overline{G}, a)}, \quad (9)$$

$$\|(P_{G,af})(z)\|_{G^\alpha(\overline{G})} \leq \mu M \|f\|_{C_{\beta-1}(\overline{G}, a)}, \quad (10)$$

$$\|(P_{G,af_1})(z) - (P_{G,af_2})(z)\|_{G_{\beta-1}(\overline{G}, a)} \leq \mu M_\beta(G) \|f_1 - f_2\|_{C_{\beta-1}(G, a)}, \quad (11)$$

are true, where  $\mu = \|A\|_{S_1(G, a)} + \|B\|_{S_1(G, a)}$ ;  $f_1, f_2$  are arbitrary functions from the class  $C_{\beta-1}(\overline{G}, a)$ .

**Proof.** Let  $f(z) \in C_{\beta-1}(\overline{G}, a)$ . Then  $f^*(z) \in S_\beta(G, a) \subset L_q(G)$ ,  $2 < q < \frac{2}{\beta}$  and the estimation

$$\|f^*\|_{S_\beta(G, a)} \leq \mu \|f\|_{C_{\beta-1}(G, a)}$$

is hold. Therefore, from (7) the inequality (9) follows. The inequality (11) can be proved similarly. As

$$\|f^*\|_{L_q(G)} \leq M \mu \|f\|_{C_{\beta-1}(G, a)}$$

inequality (10) follows from the estimation (6.8) of the work [1, ch.1, §6]. The lemma is proved.

From the form of the equation (6) and lemmas 1 and 2 by virtue of the results in [1], it follows, that any solution of this equation from the class  $C_{\beta-1}(\overline{G}, a)$  belongs to the class  $W_q^1(G)$ ,  $0 < \beta < \frac{2}{q}$ ,  $q > 2$ , and satisfies the equation (1) almost everywhere in  $G$ . Thus, the next result is true.

**Theorem 1.** *Any solution of equation (1) from the class (3) satisfies the equation (6). And vice versa, if  $\Phi(z) \in U_0(G)$ , then any solution of the equation (6) from the class  $C_{\beta-1}(\overline{G}, a)$  belongs to the class  $W_q^1(G)$ ,  $0 < \beta < \frac{2}{q}$ ,  $q > 2$ , and satisfies the equation (1) almost everywhere in  $G$ .*

Let

$$(K_\Gamma f)(z) = \frac{1}{2\pi i} \int_\Gamma \frac{f(t)}{t-z} dt, \quad (K_{\Gamma,a} f)(z) = (K_\Gamma f)(z) - (K_\Gamma f)(a).$$

Applying the operator  $(K_\Gamma f)(z)$  for  $z \in G$  to both parts of equality (6), we receive

$$(K_\Gamma V)(z) + (P_G V)(a) = (T_G F)(a) + (z-a)\Phi(z). \quad (12)$$

From here for  $z = a$  we have

$$(K_\Gamma V)(a) + (P_G V)(a) = (T_G F)(a). \quad (13)$$

From (12) and (13) it follows

$$(z-a)\Phi(z) = (K_{\Gamma,a} V)(z).$$

Hence, the integral representation of the first type for the solutions of equation (1) from the class (3) has the form

$$V(z) = -(P_{G,a} V)(z) + (K_{\Gamma,a} V)(z) + (T_{G,a} F)(z), \quad z \in G.$$

**Theorem 2.** *The equation (6) is solvable in the class  $C_{\beta-1}(\overline{G}, a)$  for any right-hand side from the same class.*

**Proof.** As  $0 < \beta < 1$ , then from  $V(z) \in C_{\beta-1}(\overline{G}, a)$  it follows, that  $V(z) \in C(\overline{G})$  and  $V(a) = 0$ . Therefore, from lemma 2 and Arzela's theorem it follows, that  $P_{G,a}$  is a completely continuous operator from  $C_{\beta-1}(\overline{G}, a)$  into  $C_{\beta-1}(\overline{G}, a) \cap C^\alpha(\overline{G})$ . Hence, for the proof of the solvability of equation (6) in class  $C_{\beta-1}(\overline{G}, a)$  it is enough to show, that the corresponding homogeneous equation

$$V(z) + (P_{G,a} V)(z) = 0, \quad z \in G \quad (14)$$

has only the trivial solution in  $C_{\beta-1}(\overline{G}, a)$ .

Let us prove now, that the homogeneous equation (14) has only the trivial solution in class  $C_{\beta-1}(\overline{G}, a)$ . The proof is performed by contradiction.

Assume that equation (14) has a not-trivial solution  $V(z)$  in the class  $C_{\beta-1}(\overline{G}, a)$ .

Applying the operator  $\partial_{\bar{z}}$  to both sides of the equation (14) we have

$$\partial_{\bar{z}}V + A(z)V + B(z)\bar{V} = 0.$$

Hence, the representation for solution

$$V(z) = \Phi(z) \exp(-\omega(z)) \quad (15)$$

holds, see [1,4], where  $\Phi(z) \in U_0(G)$ ,  $\omega(z) = (T_G V^\wedge)(z)$ ,  $V^\wedge = \frac{V^*}{V}$ .

From (15) it follows

$$\Phi(z) = V(z) \exp \omega(z) \quad (16)$$

Substituted (15) into (14) gives

$$\Phi(z) = -\exp(-\omega(z)) \cdot (P_{G,a}V)(z), \text{ where } V = \Phi(z) \exp(-\omega(z)).$$

As  $\exp(-\omega(z)) \cdot (P_{G,a}V)(z) \in U_0(E \setminus \bar{G})$ , then from the last equality it follows that the function  $\Phi(z)$  analytically single extends to the entire complex plane  $E$ .

Let  $0 < \varepsilon < 1$  and  $G_0 = \{z : |z - a| < \varepsilon\} \subset G$ . Using the Hadamard inequality, we have

$$|\omega(z)| \leq \frac{\mu}{\pi} \left( \iint_{G \setminus G_0} \frac{dG_\zeta}{|\zeta - a||\zeta - z|} - \iint_{G_0} \frac{dG_\zeta}{|\zeta - a||\zeta - z|} \right) \leq \mu(M(G \setminus G_0) + 8 \ln \frac{1}{|z - a|}),$$

where  $M(G \setminus G_0) > 0$  is a constant, depending only on  $G \setminus G_0$ . From here it follows

$$M|z - a|^{8\mu} \leq |e^{\omega(z)}| \leq \frac{M}{|z - a|^{8\mu}}, z \in G$$

Hence by virtue of (16) it follows that  $\Phi(z) \in C_{8\mu+\beta-1}(\bar{G}, a)$ .

As  $\beta > 1 - 8\mu$  if  $1 - 8\mu > 0$  and  $\beta > 0$  if  $1 - 8\mu \leq 0$  then from here it follows that  $8\mu + \beta - 1 > 0$  and  $\Phi(\infty) = 0$ .

Therefore, the function  $\Phi(z)$  analytically single extends to the entire complex plane and is equal to zero at point  $z = \infty$ . Then by Liouville theorem  $\Phi(z) \equiv 0$ . Therefore,  $V(z) \equiv 0$  in  $G$ . The theorem is proved.

From this theorem by virtue of theorem 1 the next result it follows.

**Theorem 3.** *The equation (1) is solvable in the class (3).*

## §2. Zeros and poles of solutions of equation (1)

Let  $k$  be an integer number. Let us consider the equation (1) in  $G$ , where  $F(z) \in S_{\beta-k}(G, a)$ ,  $0 < \beta < 1$ ;  $A(z), B(z) \in S_1(G, a)$ .

The solution of equation (1) from the class

$$W_q^1(G) \bigcap S_{\beta-k-1}(G, a) \quad (17)$$

is looked in the form

$$V(z) = (z - a)^k W(z), \quad (18)$$

where  $W(z)$  is a new unknown function from the class (3).

Substituted (18) into (1) we get

$$\partial_{\bar{z}} W + A(z)W + B_k(z)\overline{W} = F_k(z), \quad (19)$$

where

$$B_k(z) = B(z) \exp(-2ik\varphi), F_k(z) = (z - a)^{-k} F(z), \varphi = \arg(z - a).$$

It is obvious, that  $B_k(z) \in S_1(G, a)$ ,  $F_k(z) \in S_\beta(G, a)$ . Therefore, by virtue of the results of §1 the equation (19) has solutions from the class (3). They can be found from the equation

$$W(z) + (P_{G,a}^\wedge W)(z) = (T_{G,a} F_k)(z) + (z - a)\Phi(z), \quad (20)$$

where

$$(P_{G,a}^\wedge f)(z) = (T_{G,a} f_k^*)(z), \quad f_k^*(z) = A(z)f + B_k(z)\bar{f}.$$

Thus the next theorem is proved.

**Theorem 4.** *The equation (1), where  $F(z) \in S_{\beta-k}(G, a)$ ,  $0 < \beta < \frac{2}{q}$ ,  $q > 2$ ,  $k$  is an integer number,  $A(z), B(z) \in S_1(G, a)$ , has solutions from the class (17), which can be found by formulas (18), (20).*

### §3. Generalized Riemann-Hilbert problem for equation (1)

Let  $R > 0$ ,  $G = \{z : |z| < R\}$ ,  $\Gamma = \{t : |t| = R\}$ . Let us consider equation (1) in  $G$ , where  $A(z), B(z) \in S_1(G, 0)$ ,  $F(z) \in S_\beta(G, 0)$ ,  $0 < \beta < \frac{2}{q}$ ,  $q > 2$ .

We look for solutions of equation (1) in the class

$$W_q^1(G) \bigcap S_{\beta-1}(G, 0), \quad 0 < \beta < \frac{2}{q}, \quad q > 2. \quad (21)$$

Let us consider in  $G$  the generalized Riemann-Hilbert problem in the canonical form, see [1]. The more general cases can be reduced to this form, see [1].

**Problem R-H.** *It is necessary to find a solution of equation (1) in the class (21), satisfying the boundary condition*

$$\operatorname{Re}[t^{-m} V(t)] = g(t), \quad t \in \Gamma, \quad (22)$$

where  $m$  is an integer number,  $g(t) \in C^\alpha(\Gamma)$ ,  $0 < \beta < \frac{2}{q}$ ,  $q > 2$ .

<sup>10</sup>. Let  $m \geq 1$ . To solve the R-H problem the equation (6) is used with  $a = 0$ :

$$V + (P_{G,0} V)(z) = (T_{G,0} F)(z) + z\Phi(z), \quad z \in G, \quad (23)$$

where  $\Phi(z) \in U_0(G) \bigcap C^\alpha(\overline{G})$ ,  $\alpha = 1 - \frac{2}{q}$ .

Following [1, ch.4, §7], the function  $\Phi(z)$  in equation (23) is represented in the form

$$\Phi(z) = \Phi_0(z) - (P_m V)(z) + (Q_m F)(z), \quad (24)$$

where

$$\begin{aligned} \Phi_0(z) &\in U_0(G) \bigcap C^\alpha(\overline{G}), \quad (P_m V)(z) = (Q_m V^*)(z), \quad V^* = A(z)V + B(z)\overline{V}, \\ (Q_m f) &= -\frac{z^{2m}}{\pi R^{2m}} \iint_G \frac{\overline{f(\zeta)} dG_\zeta}{R^2 - \zeta z} - \frac{z^{2m-1}}{\pi R^{2m}} \iint_G \frac{\overline{f(\zeta)}}{\zeta} dG_\zeta. \end{aligned}$$

Substituting the representation (23) for  $V(z)$  in the boundary condition (22), we have

$$Re[t^{-m+1}\Phi_0(t)] = g(t).$$

The general solution of this problem is given by the formula, see [1, ch. 4, §7],

$$\Phi_0(z) = (D_{m-1}g)(z) + \Phi_{0m}(z), \quad (25)$$

where

$$\begin{aligned} (D_m g)(z) &= \frac{z^m}{2\pi i} \int_\Gamma g(t) \frac{t+z}{t-z} \frac{dt}{t}, \\ \Phi_{0m}(z) &= \sum_{k=0}^{m-2} (\alpha_k(z^k - R^{2(k-m-1)}z^{2m-k-2}) + i\beta_k(z^k - R^{2(k-m+1)}z^{2m-k-2})) + i\beta_m z^{m-1}, \end{aligned} \quad (26)$$

if  $m \geq 2$  and  $\Phi_{0m}(z) = i\beta_m$ , if  $m = 1$ .

Here  $\alpha_k, \beta_k$ ,  $k = 0, \dots, m-2$ ;  $\beta_m$  are arbitrary real numbers.

From formulas (23) - (25) it follows that

$$V(z) + (P_m^\wedge V)(z) = (D_m g)(z) + (Q_m^\wedge F)(z) + z\Phi_{0m}(z), \quad z \in G, \quad (27)$$

where

$$(P_m^\wedge V)(z) = (P_{G,0}V)(z) + z(P_m V)(z), \quad (Q_m^\wedge V)(z) = (T_{G,0}F)(z) + z(Q_m F)(z).$$

Thus, with  $m \geq 1$  the R-H problem is reduced to the equivalent equation (27). For any real numbers  $\alpha_k$  and  $\beta_k$ ,  $k = 0, \dots, m-2$ ;  $\beta_m$  the solution of equation (27) is the solution of the R-H problem. Let us prove, that equation (27) has a solution in the class  $C_{\beta-1}(\overline{G}, 0)$ . As in the case of the operator  $(P_{G,a}V)(z)$  it is proved, that the operator  $(P_m^\wedge V)(z)$  is completely continuous in the class  $C_{\beta-1}(\overline{G}, 0)$ .

Therefore, our statement will be proved, if we show, that the homogeneous equation

$$V + (P_m^\wedge V)(z) = 0, \quad z \in G \quad (28)$$

has no non-trivial solution in the class  $C_{\beta-1}(G, 0)$ . From (28) by virtue of the Cauchy integral formula, see [8], we get

$$(K_\Gamma V)(z) - (P_G V)(0) = -z(P_m V)(z), \quad z \in G.$$

If we compare the coefficients of the series expansions with respect to  $z$  of the left-and right-hand sides of the last equality, then we obtain

$$\int_{\Gamma} V(t) e^{-ik\theta} d\theta = 0, \quad k = 0, \dots, 2m-1; \quad t = Re^{i\theta}. \quad (29)$$

Moreover, any solution of equation (29), can be represented as

$$V(z) = \Phi(z) \exp \Omega(z), \quad (30)$$

where

$$\Phi(z) \in U_0(G) \cap C^\alpha(\overline{G}), \quad \Phi(0) = 0, \\ \Omega(z) = -\frac{1}{\pi} \iint_G \left( \frac{V^\wedge(\zeta)}{\zeta - z} - \frac{z \overline{V^\wedge(\zeta)}}{R^2 - \bar{\zeta}z} \right) dG_\zeta, \quad V^\wedge(z) = \frac{V^*(z)}{V(z)}.$$

But  $V(z)$  satisfies also the homogeneous boundary condition

$$Re[t^{-m}V(t)] = 0, \quad t \in \Gamma. \quad (31)$$

As  $Re[i\Omega(t)] = 0, t \in \Gamma$ , then the boundary condition (31) by (30) is represented as

$$Re[t^{-m}\Phi(t)] = 0.$$

The general solution of this problem with  $\Phi(0) = 0$  has the form

$$\Phi(z) = \sum_{k=1}^{2m-1} c_k z^k,$$

where  $c_k, k = 1, \dots, 2m-1$ ; are complex constants, satisfying the conditions  $c_{2m-k} = -\bar{c}_k, k = 1, \dots, m$ . Therefore, by virtue of (30) the solution of equation (28) has the form

$$V(z) = \sum_{k=1}^{2m-1} c_k z^k \exp \Omega(z).$$

Inserting this into the equality (29), we get

$$\sum_{k=1}^{2m-1} c_k \int_{\Gamma} t^k t^{-n} \exp \Omega(t) dt = 0, \quad n = 1, \dots, 2m-1. \quad (32)$$

From here follows, that  $c_k = 0, k = 1, \dots, 2m-1$ ; because the determinant of the system (32) is different from zero as the Gram determinant for the system of the linearly independent functions

$$t^k \exp \left( \frac{\Omega(t)}{2} \right), \quad k = 1, \dots, 2m-1; \quad \Omega(t) = \overline{\Omega(t)}, \quad t \in \Gamma.$$

This proves, that the homogeneous equation (28) has only the trivial solution. Hence, the integral equation (27) has a solution in  $C_{\beta-1}(\overline{G}, 0)$  for any right - hand side belonging to  $C_{\beta-1}(\overline{G}, 0)$ .



Thus for  $m > 0$  the non – homogeneous R-H problem is always solvable a solution and the homogeneous R-H problem ( $F \equiv 0$ ,  $g \equiv 0$ ) has exactly  $2m - 1$  linearly independent solutions over the field of real numbers. The latter follows from the fact that the homogeneous problem is equivalent to the integral equation

$$V + (P_m^\wedge V)(z) = z\Phi_{0m}(z),$$

where  $\Phi_{0m}(z)$  is defined by formula (26), the right-hand side of which is a linear combination of  $2m - 1$  linearly independent functions. Thus the next theorem follows.

**Theorem 5.** *For  $m > 0$  the non-homogeneous R-H problem is always solvable and the homogeneous R-H problem has  $2m - 1$  linearly independent solutions over the field of real numbers.*

2<sup>0</sup>. Let  $m = 0$ . Following to 1<sup>0</sup>, the function  $z\Phi(z)$  in the equation (23) is represented in the form

$$z\Phi(z) = (D_0g)(z) + ic_0 - z(P_0V)(z) + z(Q_0F)(z), \quad (33)$$

where  $c_0$  is an arbitrary real number.

The function  $\Phi(z)$ , given by formula (33), belongs to the class  $U_0(G)$ , if the equalities

$$(D_0g)(0) = 0, \quad (34)$$

$$c_0 = 0 \quad (35)$$

are hold.

Using (33)–(34), from (23) we obtain

$$V(z) + (P^\wedge V)(z) = (D_0g)(z) + (Q^\wedge F)(z), \quad z \in G, \quad (36)$$

where

$$(P^\wedge V)(z) = (P_{G,0}V)(z) + z(P_0V)(z), \quad (Q^\wedge V)(z) = (T_{G,0}F)(z) + z(Q_0F)(z).$$

If equality (34) holds, then any solution of equation (36) belonging to  $C_{\beta-1}(\overline{G}, 0)$  satisfies the boundary condition (22) when  $m = 0$ . Solvability of equation (36) in the class  $C_{\beta-1}(\overline{G}, 0)$  can be proved similarly to the proof of the solvability of the equation (27). Hence, under the condition (34) the R-H problem is solvable. Thus, we have the next result.

**Theorem 6.** *For the solvability of the R-H problem in the case  $m = 0$  it is necessary and sufficient that the single condition (34) is satisfied. If the condition (34) holds, then the solution of the problem can be found from equation (36).*

3<sup>0</sup>. Let  $m < 0$ . As the solutions of the R-H problem are sought from the class (21), in this case formula (27) is not suitable. Therefore, let us introduce the function  $W(z) = z^{k+1}V(z)$  into consideration, where  $k = -m$ . The function  $W(z)$  satisfies the equation

$$\partial_{\bar{z}}W + A(z)W + B_{-k-1}(z)\overline{W} = F_{-k-1}W(z),$$

where

$$B_{-k-1}(z) = \exp(2(k+1)i\varphi)B(z), \quad F_{-k-1}(z) = z^{k+1}F(z), \quad \varphi = \arg z$$

and the boundary condition

$$\operatorname{Re}[t^{-1}W(t)] = g(t), \quad t \in \Gamma.$$

It is obvious, that  $B_{-k-1}(z) \in S_1(G, 0)$ ,  $F_{-k-1}(z) \in S_\beta(G, 0)$ .

Therefore this problem corresponds to the one considered in  $1^0$  for  $m = 1$ . Hence, the function  $W(z)$  satisfies the equation

$$W(z) + (P_1^\wedge W)(z) = (D_1 g)(z) + (Q_1^\wedge F_{-k-1})(z) + ic_0 z, \quad z \in G,$$

where  $c_0$  is an arbitrary real number,

$$(P_1^\wedge W)(z) = (T_{G,0}W_{-k-1}^*)(z) + z(Q_1 W_{-k-1}^*)(z), \quad W_{-k-1}^* = A(z)W + B_{-k-1}(z)\overline{W}.$$

Thus, when  $m < 0$  the solution of the R-H problem can be found from the equation

$$V(z) + (QV)(z) = \frac{z}{\pi i} \int_{\Gamma} \frac{g(t)dt}{t^{k+1}(t-z)} + (T_{G,0}F_{-k-1})(z) + \sum_{j=1}^k a_j z^{-j}, \quad z \in G, \quad (37)$$

where

$$(QV)(z) = (T_{G,0}(z^{k+1}V^*(z)))(z) - \frac{z}{\pi} \iint_{\Gamma} \frac{\bar{\zeta}^{2k} \overline{V^*(\zeta)}}{1 - \bar{\zeta}z} dG_{\zeta},$$

$$a_j = \frac{1}{\pi} \iint_G \zeta^{j-1} V^*(\zeta) dG_{\zeta} + \frac{1}{\pi} \iint_G \bar{\zeta}^{2k-j-1} \overline{V^*(\zeta)} dG_{\zeta} + \frac{1}{\pi i} \int_{\Gamma} t^{j-k-1} g(t) dt, \quad g = 0, \dots, k-1;$$

$$a_k = \frac{1}{\pi} \iint_G \zeta^{k-1} V^*(\zeta) dG_{\zeta} + \frac{1}{2\pi i} \iint_{\Gamma} \frac{g(t)}{t} dt + ic_0, \quad V^* = A(z)V + B(z)\overline{V}.$$

From (37) it follows that for the continuity of the function  $V(z)$  inside of  $G$  it is necessary and sufficient that the equalities

$$a_j = 0, \quad j = 0, \dots, k; \quad (38)$$

are hold. The condition (38) contain  $2k + 2$  real equalities. One of them, namely  $\operatorname{Im} a_k = 0$ , is possible to be satisfied by means of a suitable choice of the constant  $c_0$ . Hence, there remain  $2k + 1$  conditions. Thus, for  $m < 0$  the R-H problem is reduced to the equation

$$V(z) + (QV)(z) = \frac{z}{\pi i} \int_{\Gamma} \frac{g(t)dt}{t^{k+1}(t-z)} + (T_{G,0}F_{-k-1})(z), \quad z \in G. \quad (39)$$

The operator  $Q$  is completely continuous in  $C_{\beta-1}(G, 0)$  and mapping this space into  $C_{\beta-1}(\overline{G}, 0) \cap C^\alpha(\overline{G})$ .

If in the equation

$$V(z) + (QV)(z) = 0$$

we replace  $V(z)$  by  $zW(z)$ , then we obtain the equation (7.33) from [1, ch.4, §7], which has only the trivial solution. Therefore equation (39) is solvable in  $C_{\beta-1}(G, 0)$  for any right-hand side from the same class. Thus, we have the next result.

**Theorem 7.** *For the solvability of the R-H problem in the case  $m < 0$  it is necessary and sufficient that the  $2|m| + 1$  real conditions (38) are satisfied.*

#### §4. Riemann-Hilbert problem with an initial condition for equation (1)

Let  $G = \{z : |z| < R\}$ ,  $\Gamma = \{t : |t| = R\}$ ,  $\nu > 0$ ,  $k = [\nu]$ ,  $k \neq \nu$ ,  $\beta = 1 - \nu + k$ ,  $R > 0$ . Let us consider the equation (1) in  $G$ , where  $A(z)$ ,  $B(z) \in S_1(G, 0)$ ,  $F(z) \in S_{1-\nu}(G, 0)$ .

Let us consider the Riemann-Hilbert problem with an initial condition in the following form.

**Problem  $(R - H)_0$ .** *Find the solution of the equation (1) from the class*

$$C_{-\nu}(G, 0) \cap W_q^1(G), \quad 2 < q < \frac{2}{\beta},$$

*satisfying the boundary condition*

$$\operatorname{Re}[t^{-n}V(t)] = g(t), \quad t \in \Gamma, \quad (40)$$

where  $g(t) \in C^\alpha(\Gamma)$ ,  $\alpha = 1 - \frac{2}{q}$ ,  $n$  is an integer number.

The solution of the  $(R - H)_0$  problem is looked for in the form

$$V(z) = z^k W(z), \quad (41)$$

where  $W(z)$  is a new unknown function from the class

$$W_q^1(G) \cap S_{\beta-1}(G, 0), \quad 0 < \beta < \frac{2}{q}, \quad q > 2.$$

**Remark 1.** *If  $\nu < 1$ , the substitution (41) is not required.*

**Remark 2.** *If  $\nu > 0$  is an integer number, then from  $V(z) = O(|z|^{\nu+\beta})$ ,  $z \rightarrow 0$ ,  $0 < \beta < 1$ , it follows that  $V(z) = O(|z|^\nu)$ ,  $z \rightarrow 0$ . Therefore, the results which will be obtained for  $[\nu] \neq \nu$ , will also hold for  $[\nu] = \nu$ .*

Substituting (41) into (1) and (40), respectively, we obtain

$$\partial_{\bar{z}}W + A(z)W + B_k(z)\overline{W} = F_k(z), \quad z \in G, \quad (42)$$

where

$$B_k(z) = B(z) \cdot \exp(-2ik\varphi), \quad F_k(z) = z^{-k}F(z)$$

and

$$\operatorname{Re}[t^{-m}W(t)] = g(t), \quad t \in \Gamma, \quad m = n - k.$$

It is obvious, that  $B_k(z) \in S_1(G, 0)$ ,  $F_k(z) \in S_\beta(G, 0)$ ,  $0 < \beta < 1$ . Hence, we obtain the Riemann -Hilbert problem solved in §3 for the equation (42). Therefore, from the results of §3 the next result follows.

**Theorem 8.** 1) For  $n > [\nu]$  the problem  $(R - H)_0$  is always solvable. The corresponding homogeneous problem has  $2(n - [\nu]) - 1$  linearly independent solutions over the field of real numbers.

2) For  $n = [\nu]$  for the solvability of the  $(R - H)_0$  problem it is necessary and sufficient that the single condition (34) is satisfied.

3) For  $n < [\nu]$  for the solvability of the  $(R - H)_0$  problem it is necessary and sufficient that  $2n - [\nu] + 1$  conditions of the type (38) are satisfied, which are written with respect to the equation (42).

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